Monmouth COLLEGE

- Name: ______
- Date: ______
- Section: ______

ECON 300: Intermediate Price Theory

Problem Set #5: Suggested Solutions

INSTRUCTIONS:

- This problem set is not graded.

Problem 1. Cobb-Douglas Production Functions

Suppose that you are producing based on the following technology:

$$F(L, K) = 3L^2K$$

1.A Find the Marginal Product of Labor.

Solution:

$$MP_L = \frac{\partial F(L, K)}{\partial L} = \frac{\partial}{\partial L} 3L^2K = 6LK$$

1.B Find the Marginal Product of Capital.

Solution:

$$MP_K = \frac{\partial F(L, K)}{\partial K} = \frac{\partial}{\partial K} 3L^2K = 3L^2$$

1.C Find the Marginal Rate of Technical Substitution.

Solution:

$$MRTS_{LK} = \frac{MP_L}{MP_K} = \frac{6LK}{3L^2} = \frac{2K}{L}$$

1.D Complete the following statement:

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If the firm reduces their input of L by 1 unit,
given that the firm keeps output levels constant,
they must increase K by  2K/L  units.
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1.E What is the output level when $L = 5$ and $K = 10$?

Solution:

$$F(5, 10) = 3 \cdot (5)^2 \cdot (10) = 3 \cdot 25 \cdot 10 = 750$$

1.F What is the output level when $L = 10$ and $K = 20$?

Solution:

$$F(10, 20) = 3 \cdot (10)^2 \cdot (20) = 3 \cdot 25 \cdot 10 = 6,000$$

1.G Does this production technology display constant returns to scale? Why?

Solution:

Compring the answers of 1.E and 1.F, when we increased input by a factor of two, output increased by a factor greater than 2. This is characteristic of a production technology exhibiting Increasing Returns to Scale.

1.H Suppose that the firm's production function is updated to $F(L, K) = 3L^2K^2$. Is this representative of technical progress? Why?

Solution:

If the input of capital remains above 1, this change represents technical progress. Suppose that the original production function is $F_0(L, K)$, and the updated production function is $F_1(L, K)$:

$$\frac{F_1(L, K)}{F_0(L, K)} = \frac{3L^2K^2}{3L^2K} = K$$

When $K > 1$, $F_1(L, K) > F_0(L, K)$, and we have technical progress.

Problem 2. Short Run and Long Run Costs

Suppose that you are aiming to produce 100 units of output using inputs of labor L and capital K. Due to congestion in the supply chain, the amount of capital input you have access to is limited to (and fixed at) 5. The prevailing wage for 1 unit of L is given as $w = 5$, and the rent for 1 unit of capital is given as $r = 10$. Suppose further that you are producing based on the following technology:

$$F(L, K) = 2LK$$

2.A Find the Short Run Conditional Factor Demand.

Solution:

Use the production function, pre-populate it with the amount of fixed capital $\bar{K} = 5$, set it to the production quota $Q = 100$, and solve for L.

$$F(L, 5) = 2 \cdot L \cdot 5 = 100 \quad \Rightarrow \quad L^* = 10$$

2.B What is the value of the Short Run Total Cost if the only costs are labor and capital?

Solution:

Short run total cost is simply "wage times labor input plus rent times capital input." Which can be expressed as $STC(Q) = wL + rK$.

$$STC(100) = 5 \cdot 10 + 10 \cdot 5 = 100$$

Starting from question 2.C and beyond, suppose that the supply chain congestion is resolved, and you may freely change the level of capital input in your production. All other variables remain constant. Your target output remains at 100 units, wage is 5, and rent is 10.

2.C Find the Marginal Rate of Technical Substitution between Labor and Capital.

Solution:

$$MRTS_{LK} = \frac{MP_L}{MP_K} = \frac{2K}{2L} = \frac{K}{L}$$

2.D Find the optimal ratio of Labor to Capital.

Solution:

$$MRTS_{LK} = \frac{w}{r} \quad \Rightarrow \quad \frac{K}{L} = \frac{5}{10} \quad \Rightarrow \quad 2K = L$$

2.E Express the firm's Isocost as a mathematical equation.

Solution:

When wages are w, rent is r, units of labor employed is L, and units of capital rented is K, the isocost representing a cost level C can be expressed as:

$$wL + rK = C$$

2.F Find the optimal inputs of labor and capital.

Solution:

Use the optimal ratio found in 2.D, and the production function and quota to find the optimal inputs:

$$\begin{aligned} F(L,K) = 100 \quad &\Rightarrow \quad 2LK = 100 \\ &\Rightarrow \quad 2(2K)K = 100 && \because 2K = L \text{ from 2.D.} \\ &\Rightarrow \quad 4K^2 = 100 \\ &\Rightarrow \quad K^2 = 25 \\ &\Rightarrow \quad K^* = 5 && \because K > 0 \end{aligned}$$

We can now insert the optimal units of capital into the optimal ratio equation from 2.D:

$$2K = L \quad \Rightarrow \quad 2 \cdot 5 = L \quad \Rightarrow \quad L^* = 10$$

Problem 3. Cost Functions

Suppose that you are chosen as the supplier of Monmouth College logo embedded diploma frames. In your woodworking studio, you can hire workers (L) that can work with lumber (K) to produce diploma frames (Q). The market wage is $w = 20$ and the price of each unit of lumber is $r = 10$. Your current production technology can be described as:

$$F(L, K) = K^{\frac{1}{2}} L^{\frac{1}{2}}$$

3.A What is the optimal level of L and K if the target output was $Q = 100$?

Solution:

First find the optimal ratio between capital and labor by calculating the $MRTS$ and setting it equal to the relative input prices:

$$MRTS_{LK} = \frac{w}{r} \quad \Rightarrow \quad \frac{\frac{1}{2}K^{\frac{1}{2}}L^{-\frac{1}{2}}}{\frac{1}{2}K^{-\frac{1}{2}}L^{\frac{1}{2}}} = \frac{20}{10} \quad \Rightarrow \quad \frac{K}{L} = 2 \quad \Rightarrow \quad 2L = K$$

Use this optimal ratio and the production quota to find the optimal L and K:

$$K^{\frac{1}{2}}L^{\frac{1}{2}} = 100 \quad \Rightarrow \quad (2L)^{\frac{1}{2}}L^{\frac{1}{2}} = 100 \quad \Rightarrow \quad \sqrt{2}L = 100 \quad \Rightarrow \quad L^* = \frac{100}{\sqrt{2}}$$

The optimal amount of capital can be found using the optimal ratio:

$$2L = K \quad \Rightarrow \quad 2 \cdot \frac{100}{\sqrt{2}} = K \quad \Rightarrow \quad K^* = \frac{200}{\sqrt{2}}$$

3.B What is the optimal level of L and K for an arbitrary level of output Q?

Solution:

Use the optimal ratio found in 3.A and the production quota to find the optimal L and K:

$$K^{\frac{1}{2}}L^{\frac{1}{2}} = Q \quad \Rightarrow \quad (2L)^{\frac{1}{2}}L^{\frac{1}{2}} = Q \quad \Rightarrow \quad \sqrt{2}L = Q \quad \Rightarrow \quad L^* = \frac{Q}{\sqrt{2}}$$

The optimal amount of capital can be found using the optimal ratio:

$$2L = K \quad \Rightarrow \quad 2 \cdot \frac{Q}{\sqrt{2}} = K \quad \Rightarrow \quad K^* = \frac{2Q}{\sqrt{2}}$$

3.C Find the Total Cost function for producing diploma frames. *(Hint: $TC(Q) = w \cdot L + r \cdot K$)*

Solution:

$$TC(Q) = wL + rK = 20 \cdot \frac{Q}{\sqrt{2}} + 10 \cdot \frac{2Q}{\sqrt{2}} = \frac{20}{\sqrt{2}}Q + \frac{20}{\sqrt{2}}Q = \frac{40}{\sqrt{2}}Q$$

3.D Find the Average Total Cost function, $ATC(Q)$.

Solution:

$$ATC(Q) = \frac{TC(Q)}{Q} = \frac{40}{\sqrt{2}}$$

3.E Find the Marginal Cost function, $MC(Q)$.

Solution:

$$MC(Q) = \frac{dTC(Q)}{dQ} = \frac{40}{\sqrt{2}}$$

3.F Complete the chart below by plotting the three functions $TC(Q)$, $ATC(Q)$, and $MC(Q)$.