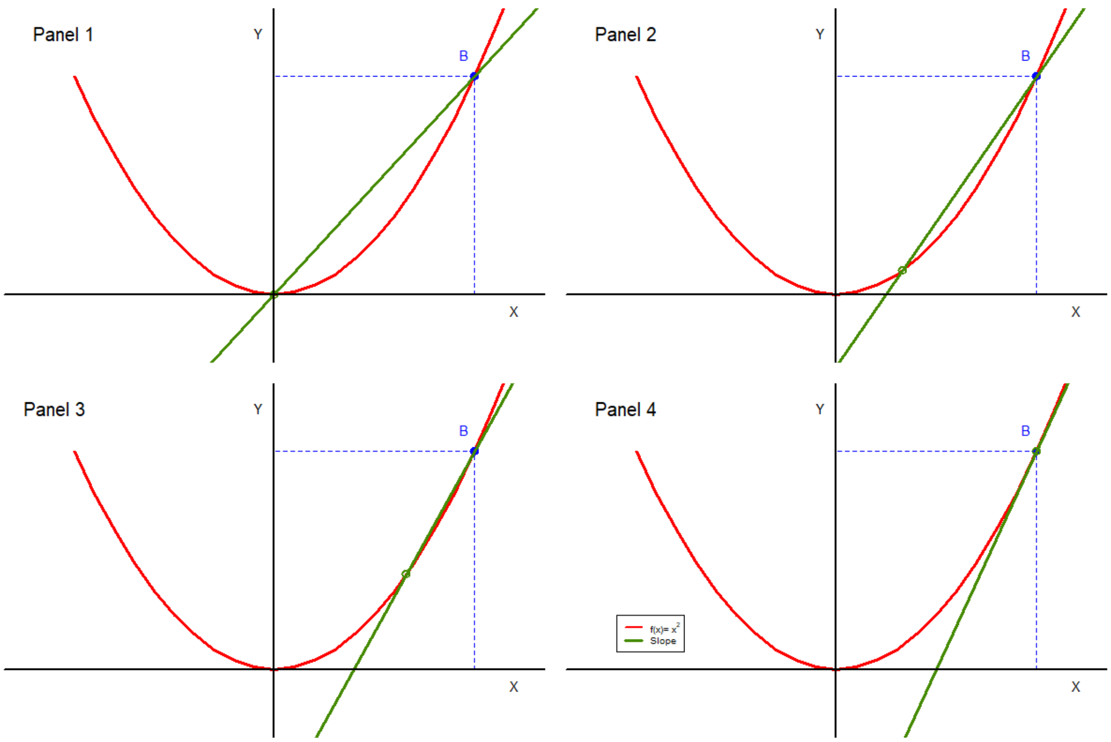
Handout #3: Derivatives

ECON 300: Intermediate Price Theory

Topic 1. Derivatives

One approach to address the challenge of selecting two points is to attempt to determine the slope of a function at a singular point, rather than two arbitrary points. This won't alter the reality that the slope fluctuates along the chosen single point, but it will help simplify matters by eliminating one choice entirely. Let's revisit the quadratic case from handout #2 and explore the possibilities.



Observe that while transitioning from panels 1 to 4, we are shifting the "end point" (green hollow dot) from the origin towards point B. If we can accomplish this, we can mitigate some of the complexity inherent in calculating slopes for non-linear functions. In fact, Panel 4 portrays what is known as "the first derivative of $f(x) = x^2$ evaluated at point $x = 1$."

Formally, the first derivative of $f(x)$ with respect to x is defined as:[1]

$$\frac{d}{dx}f(x) = f'(x) \equiv \lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - x} \tag{1}$$

[1] We will ignore the conditions of differentiability, as it is beyond the scope of this course.

Being less rigorous for the sake of simplicity, we can also say:

$$\frac{d}{dx}f(x) = f'(x) \simeq \frac{f(x) - f(a)}{x - a}\text{, given that } x \text{ and } a \text{ are incredibly close.}$$

Topic 2. Differentiation Rules

The most crucial fundamental rule to grasp when delving into derivatives is the power rule:

- $\frac{d}{dx}x^a = ax^{a-1}$ Power Rule

Now suppose that there are two functions $f(x)$ and $g(x)$ that are each continuously differentiable, and that c is a constant.[2] With these premises, the following rules apply:

- $\frac{d}{dx}c = 0$ Derivative of Constants

- $\frac{d}{dx}\left[f(x) + g(x)\right] = f'(x) + g'(x)$ Derivative of a Sum

- $\frac{d}{dx}f(x)g(x) = f'(x)g(x) + f(x)g'(x)$ Product Rule

- $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{\left[g(x)\right]^2}$ Quotient Rule

- $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$ Chain Rule

Among the rules mentioned above, the first three are particularly relevant within the context of this course: the power rule, the derivative of a constant, and the derivative of a sum. While the remaining rules can occasionally be of use, I will omit them for now.

[2] For now, ignore the "continuously differentiable" part, as it is not within the scope of this course.

Please try out using the rules above to find the first derivative of the following functions:

6. $f(x) = -3x^9$

7. $f(x) = 2x^{-1}$

8. $f(x) = 5x$

9. $f(x) = 3$

10. $f(x) = x^6 + 8x - 19$

11. $f(x) = 2x^{\frac{1}{2}} + 3$

Topic 5. Why: The Technical Aspect

Remember that we introduced the formal definition of a derivative in equation (1). It is *technically* feasible to determine the derivative of certain functions using the definition itself. For example, let's say you wish to find the first derivative of the function $f(x) = x^2 + 3x + 1$:

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{\{(x+h)^2 + 3(x+h) + 1\} - x^2 - 3x - 1}{(x+h) - x} \\
&= \lim_{h \to 0} \frac{x^2 + 2hx + h^2 + 3x + 3h + 1 - x^2 - 3x - 1}{h} \\
&= \lim_{h \to 0} \frac{2hx + h^2 + 3h}{h} \\
&= \lim_{h \to 0} \frac{h(2x + h + 3)}{h} \\
&= \lim_{h \to 0} (2x + h + 3) \\
&= 2x + 3
\end{aligned}
$$

It's evident why utilizing the differentiation rules is a more straightforward approach, especially when we need to replicate this process numerous times to arrive at solutions for various inquiries in ECON 300.

Topic 6. Why: Yeah... Okay... But WHY?

One of the fundamental concepts that underpins much of orthodox economic theory is "marginalism." This principle is frequently associated with Alfred Marshall and his 1890 publication *Principles of Economics*.[3] When confronted with a choice between alternatives, economists analyze at the margin. This often entails comparing the benefits and costs associated with the final unit we consume or produce.

So, why is marginal analysis highly valuable in the realm of economics? It's because *when the marginal value of a variable transitions from positive to negative, the variable itself is maximized*.[4] Let's assume you have the following profit data:

Quantity	Profit	Average Profit	"Marginal" Profit
10	$500	$50	N/A
20	$1,200	$60	$700
30	$1,700	$56.6	$500
40	$2,000	$50	$300
50	$2,100	$42	$100
60	$2,000	$33.3	-$100
70	$1,700	$24.3	-$300
80	$1,200	$15	-$500
90	$500	$5.6	-$700

At first, it might seem intuitive to assume that profits are maximized when the average profit per unit reaches its peak. However, this is rarely the situation, as we will uncover when we delve into producer theory. Although this is a simplified example, it demonstrates the validity of the principle. As the marginal profit shifts from a positive value to a negative value at a production quantity of 50, it becomes evident that the profit is indeed maximized.

To see this in greater detail, let us examine the underlying profit function. The profit function for this case is:

$$\Pi(Q) = -Q^2 + 100Q - 400$$

Then the average profit is:

$$\frac{\Pi(Q)}{Q} = -Q + 100 - \frac{400}{Q}$$

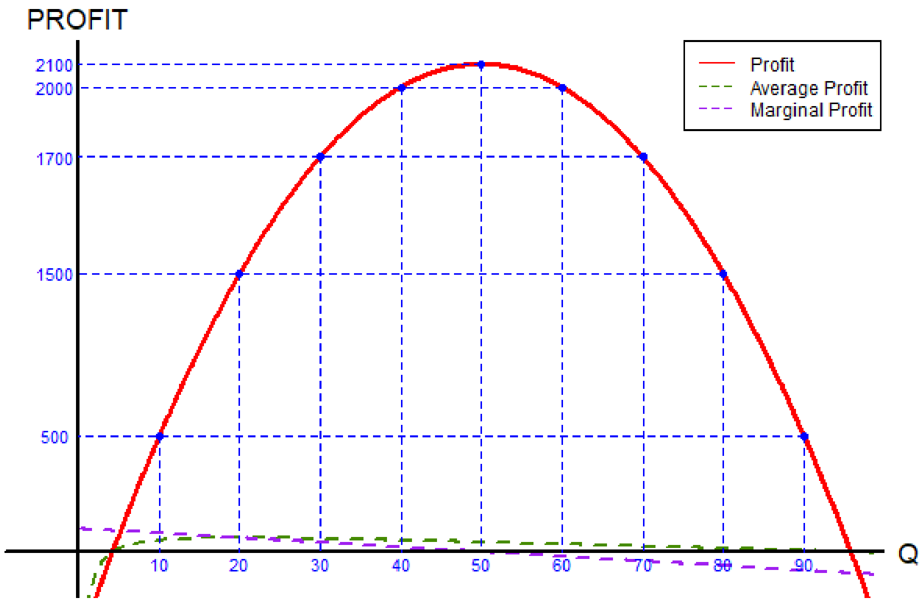
Finally, the marginal profit is:

$$\frac{d\Pi(Q)}{dQ} = -2Q + 100$$

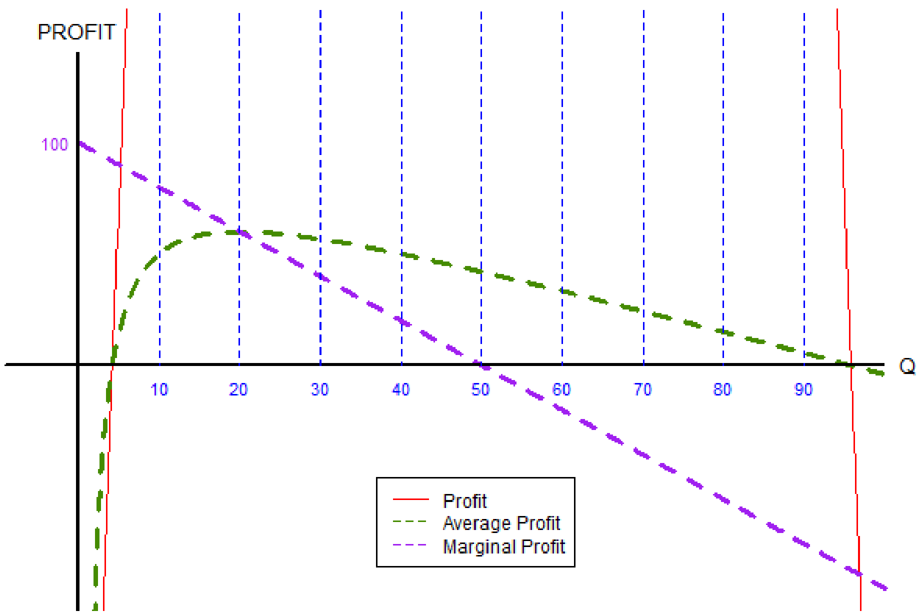
[3]Some credit Léon Walras as a pioneer of marginalist theories in the field of economics.

[4]It could also be minimized, but we'll defer that discussion for now.

Plotting these three functions, we have:



Notice that the peak of the profit function is actually achieved at $Q = 50$ for a profit of $2,100. Now, let's adjust the graph's scale to allow for a clearer visualization of the average and marginal profit curves.



It's evident that the marginal cost curve dips into negative values at the $Q = 50$ point, further validating the initial assertion that *when the marginal value of a variable transitions from positive to negative, the variable itself is maximized*.