



• Name: \_\_\_\_\_

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## **BUSN 301: Intermediate Microeconomic Theory**

### **Problem Set #4: Suggested Solutions**

**Spring 2026**

#### **INSTRUCTIONS:**

- Each problem set is graded on a 100-point basis and contributes to your Problem Set component of the course grade.
- You are expected to show all relevant steps and reasoning.
- Answers must be clearly written and well-organized.
- Graphs, when required, must be clearly labeled, with axes, curves, and key points identified.
- Problem sets must be submitted by the posted deadline.

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**Problem 1. Exchange**

Consider a pure exchange economy with two agents, A and B, and two goods, 1 and 2. Each agents' utility function is given by:

$$u_A(x_A^1, x_A^2) = (x_A^1)^{\frac{1}{2}}(x_A^2)^{\frac{1}{2}}, \quad u_B(x_B^1, x_B^2) = (x_B^1)^{\frac{1}{4}}(x_B^2)^{\frac{3}{4}}$$

The initial endowment of the economy is given by  $(\omega_A^1, \omega_A^2, \omega_B^1, \omega_B^2) = (4, 2, 2, 6)$ .

1.A. State the utility maximization problem for agent A. Clearly define the objective function and their budget constraint.

$$\max_{x_A^1, x_A^2} (x_A^1)^{\frac{1}{2}}(x_A^2)^{\frac{1}{2}}, \quad \text{s.t.} \quad p_1x_A^1 + p_2x_A^2 = 4p_1 + 2p_2$$

1.B. Solve agent A's utility maximization problem to obtain their demand for goods 1 and 2 as functions of prices  $(p_1, p_2)$  and income.

The first-order condition for agent A is:

$$MRS = \frac{p_1}{p_2} \Rightarrow \frac{\frac{1}{2}(x_A^1)^{-\frac{1}{2}}(x_A^2)^{\frac{1}{2}}}{\frac{1}{2}(x_A^1)^{\frac{1}{2}}(x_A^2)^{-\frac{1}{2}}} = \frac{p_1}{p_2} \Rightarrow \frac{x_A^2}{x_A^1} = \frac{p_1}{p_2} \Rightarrow x_A^2 = \frac{p_1}{p_2}x_A^1$$

Substituting the optimal ratio into the agent's budget constraint:

$$\begin{aligned} p_1x_A^1 + p_2x_A^2 = 4p_1 + 2p_2 &\Rightarrow p_1x_A^1 + p_2\left(\frac{p_1}{p_2}x_A^1\right) = 4p_1 + 2p_2 \\ &\Rightarrow 2p_1x_A^1 = 4p_1 + 2p_2 \\ &\Rightarrow x_A^1 = \frac{4p_1 + 2p_2}{2p_1} \end{aligned}$$

Substituting the optimal ratio into the agent's budget constraint, we find:

$$x_A^2 = \frac{p_1}{p_2}x_A^1 \Rightarrow x_A^2 = \frac{p_1}{p_2} \cdot \frac{4p_1 + 2p_2}{2p_1} \Rightarrow x_A^2 = \frac{4p_1 + 2p_2}{2p_2}$$

1.C. Repeat parts 1 . A and 1 . B for agent B

Repeating the steps above, we find that:

$$x_B^1 = \frac{2p_1 + 6p_2}{4p_1}, \quad x_B^2 = \frac{3(2p_1 + 6p_2)}{4p_2}$$

**Problem 1. Exchange (continued)**

1.D. Let good 2 be the numeraire good, so that  $p_2 = 1$ , and let  $p_1 = p$ . Compute each agent's income as a function of  $p$ .

Using the budget constraint and substituting  $p_2 = 1$  and  $p_1 = p$ , we obtain:

$$\text{Agent A : } p_1\omega_A^1 + p_2\omega_A^2 \Rightarrow \boxed{4p + 2}, \quad \text{Agent B : } p_1\omega_B^1 + p_2\omega_B^2 \Rightarrow \boxed{2p + 6}$$

1.E. Substitute income into each agent's demand for good 1. Express demand as a function of  $p$ .

Agent A's tangency condition is:

$$MRS = p \Rightarrow \frac{x_A^2}{x_A^1} = p \Rightarrow x_A^2 = px_A^1$$

Substituting into the budget constraint using income from 1 . D:

$$px_A^1 + x_A^2 = 4p + 2 \Rightarrow 2px_A^1 = 4p + 2 \Rightarrow \boxed{x_A^1 = \frac{4p + 2}{2p}} \Rightarrow \boxed{x_A^2 = \frac{4p + 2}{2}}$$

Agent B's demand for goods 1 and 2 are

$$\boxed{x_B^1 = \frac{2p + 6}{4p}, \quad x_B^2 = \frac{3(2p + 6)}{4}}$$

1.F. Find the competitive equilibrium price  $p^*$ , and the equilibrium allocation.

The equilibrium price is the price that clears the market for good 1:

$$x_A^1 + x_B^1 = \omega_A^1 + \omega_B^1 \Rightarrow \frac{4p^* + 2}{2p^*} + \frac{2p^* + 6}{4p^*} = 6 \Rightarrow \frac{5p^* + 5}{2p^*} = 6 \Rightarrow \boxed{p^* = \frac{5}{7}}$$

Substituting  $p^*$  back into each agent's demand:

$$\boxed{(x_A^1, x_A^2, x_B^1, x_B^2) = \left(\frac{17}{5}, \frac{17}{7}, \frac{13}{5}, \frac{39}{7}\right)}$$

1.G. Verify that the equilibrium is Pareto optimal (brief explanation).

The equilibrium allocation is Pareto optimal if  $MRS_A = MRS_B$ . At a competitive equilibrium, each agent  $i$  chooses a bundle such that  $MRS_i = p$ . Since both agents face the same price ratio, we have  $MRS_A = MRS_B$ . Therefore, the competitive equilibrium allocation is Pareto optimal.

**Problem 2. Production**

Consider an economy with one consumer, one firm, and two goods, 1 and 2. The consumer's initial endowment is given by  $(\omega^1, \omega^2) = (0, 12)$ . The consumer's utility function is given by

$$u(x^1, x^2) = (x^1)^{\frac{1}{2}}(x^2)^{\frac{1}{2}}.$$

The firm uses good 2 as an input to produce good 1 according to the production function

$$y = f(z) = z^{\frac{1}{2}},$$

where  $z$  denotes the amount of good 2 used as input, and  $y$  denotes the amount of good 1 produced. Assume that the consumer owns the firm.

2.A. State the firm's profit maximization problem. Clearly define the firm's profit function.

Since the price of a unit of output is  $p_1$ , and the cost of a unit of input is  $p_2$ :

$$\max_z \pi = p_1 z^{\frac{1}{2}} - p_2 z$$

2.B. Solve the firm's profit maximization problem to find its demand for input  $z$ , output supply  $y$ , and profit as functions of prices  $(p_1, p_2)$ .

Following the first-order condition for profit maximization:

$$\frac{d\pi}{dz} = 0 \Rightarrow \frac{1}{2}p_1(z^*)^{-\frac{1}{2}} - p_2 = 0 \Rightarrow z^* = \frac{p_1^2}{4p_2^2}$$

Using the production function we find output, and substituting into the profit function:

$$y^* = (z^*)^{\frac{1}{2}} \Rightarrow y^* = \frac{p_1}{2p_2}, \quad \pi = p_1 y - p_2 z \Rightarrow \pi = \frac{p_1^2}{4p_2}$$

2.C. State the consumer's utility maximization problem. Clearly define the consumer's budget constraint.

$$\max_{x^1, x^2} u(x^1, x^2) \quad \text{s.t.} \quad p_1 x^1 + p_2 x^2 = p_1 \omega^1 + p_2 \omega^2 + \pi$$

Substituting in the known values and expressions:

$$\max_{x^1, x^2} (x^1)^{\frac{1}{2}}(x^2)^{\frac{1}{2}} \quad \text{s.t.} \quad p_1 x^1 + p_2 x^2 = 12p_2 + \pi$$

**Problem 2. Production (continued)**

2.D. Let good 2 be the numeraire good, so that  $p_2 = 1$ , and let  $p_1 = p$ . Express the consumer's income as a function of  $p$ , taking into account both the value of the initial endowment and firm profit.

Applying  $p_2 = 1$ ,  $p_1 = p$ , and the initial endowment  $(\omega^1, \omega^2) = (0, 12)$ , we find:

$$p_1\omega^1 + p_2\omega^2 + \pi \Rightarrow p \cdot 0 + 1 \cdot 12 + \pi \Rightarrow 12 + \pi$$

Substituting the firm's profit function from 2 . B, the consumer's income is:

$$12 + \pi \Rightarrow \boxed{12 + \frac{p^2}{4}}$$

2.E. Solve the consumer's utility maximization problem to find the consumer's demand for good 1 as a function of  $p$ .

The tangency condition for the consumer is:

$$MRS = p \Rightarrow \frac{x^2}{x^1} = p \Rightarrow x^2 = px^1$$

Substituting into the budget constraint:

$$px^1 + x^2 = 12 + \frac{p^2}{4} \Rightarrow px^1 + px^1 = 12 + \frac{p^2}{4} \Rightarrow \boxed{x^1 = \frac{p^2 + 48}{8p}}$$

2.F. Find the competitive equilibrium price  $p^*$ .

We find the competitive equilibrium price by applying market clearing conditions. We find the firm's supply from 2 . B, and the consumer's demand from 2 . E:

$$y(p^*) = x^1(p^*) \Rightarrow \frac{p^*}{2} = \frac{(p^*)^2 + 48}{8p^*} \Rightarrow 3(p^*)^2 = 48 \Rightarrow \boxed{p^* = 4}$$

**Problem 2. Production (continued)**

2.G. Compute the competitive equilibrium allocation and the firm's equilibrium production plan.

The consumer's allocation can be found by substituting  $p^*$  into the demand function from 2.E:

$$x^1(p^*) = \frac{(p^*)^2 + 48}{8p^*} \Rightarrow \boxed{x^1 = 2}$$

From the optimal ratio found in 2.E:

$$x^2(p^*) = p^* x^1(p^*) \Rightarrow \boxed{x^2 = 8}$$

The producer's allocation can be found by substituting  $p_1 = p^*$  and  $p_2 = 1$  into the results found in 2.B:

$$z^* = \frac{p_1^2}{4p_2^2} \Rightarrow \boxed{z^* = 4}, \quad y^* = \frac{p_1}{2p_2} \Rightarrow \boxed{y^* = 2}$$

2.H. Verify that, at equilibrium, the consumer's marginal rate of substitution equals the marginal rate of transformation.

The consumer's marginal rate of substitution at the optima is:

$$MRS = \frac{x^2}{x_1} = \frac{8}{2} = 4$$

The producer's marginal rate of transformation at the optima is:<sup>1</sup>

$$MRT = \frac{dz}{dy} = \frac{dy^2}{dy} = 2y = 4$$

At the optimal allocation, we find that  $MRS = MRT$ .

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<sup>1</sup>Rewrite the production function  $y = z^{\frac{1}{2}}$  to  $z = y^2$ .

**Problem 3. Welfare**

Consider the pure exchange economy described in Problem 1.

3.A. Define what it means for an allocation to be Pareto optimal.

- An allocation is Pareto optimal if there are no feasible allocations that make at least one agent strictly better off without making any other agent worse off.
- Equivalently, an allocation is Pareto optimal if there are no Pareto improvements available in the economy.

3.B. Define what it means for an allocation to be envy-free.

- An allocation is envy-free (equitable) if no agent prefers another agent's consumption bundle to their own.
- That is, for all agents  $i$  and  $j$ , we have

$$u_i(x_i) \geq u_i(x_j).$$

3.C. Define what it means for an allocation to be fair.

- An allocation is fair if it is both Pareto optimal and equitable (envy-free).
- That is, the allocation is efficient and no agent envies another agent's bundle.

3.D. Is the competitive equilibrium allocation from Problem 1 guaranteed to be Pareto optimal? Briefly explain.

- Yes. By the First Fundamental Theorem of Welfare Economics, any competitive equilibrium allocation is Pareto optimal, provided that preferences are locally non-satiated.
- In this economy, both agents have strictly monotonic preferences, so the competitive equilibrium allocation from Problem 1 is Pareto optimal.

**Problem 3. Welfare (continued)**

3.E. Suppose instead that the initial endowment is equally split:  $(\omega_A^1, \omega_A^2, \omega_B^1, \omega_B^2) = (3, 4, 3, 4)$ . Would the competitive equilibrium allocation be considered fair? Briefly explain.

- Yes. When the initial endowment is equally split, each agent begins with the same bundle.
- By the First Fundamental Theorem of Welfare Economics, the competitive equilibrium allocation is Pareto optimal.
- In addition, when each agent starts with the same endowment, the competitive equilibrium allocation is envy-free.
- Therefore, the competitive equilibrium allocation is fair.

3.F. Consider the social welfare function  $W(u_A, u_B) = u_A + u_B$ . Describe the problem faced by a social planner. What condition must hold at the optimal allocation?

- A social planner chooses a feasible allocation to maximize total social welfare:

$$\begin{aligned} \max_{x_A^1, x_A^2, x_B^1, x_B^2} \quad & u_A(x_A^1, x_A^2) + u_B(x_B^1, x_B^2) \\ \text{s.t.} \quad & x_A^1 + x_B^1 = \omega_A^1 + \omega_B^1 \\ & x_A^2 + x_B^2 = \omega_A^2 + \omega_B^2 \end{aligned}$$

- At an interior optimum, the allocation must be Pareto efficient, so the agents' marginal rates of substitution must be equal:  $MRS_A = MRS_B$ .

3.G. Briefly explain the Second Welfare Theorem and its economic intuition.

- The Second Welfare Theorem states that, under appropriate assumptions, any Pareto optimal allocation can be decentralized as a competitive equilibrium after a suitable redistribution of initial endowments.
- The intuition is that society can separate efficiency from equity: first redistribute endowments to achieve a desired distribution, and then allow competitive markets to generate an efficient allocation.

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